Maths for Computing
Tutorial 7

1. A traveling agent has to visit four cities, each of them five times. In how many different ways can he do this if he is not allowed to start and finish in the same city?

2. A host invites n couples to a party. She wants to ask a subset of the $2n$ guests to give a speech, but she does not want to ask both members of any couple to give speeches. In how many ways can she proceed?

3. How many ways are there to select an 11-member soccer team and a 5-member basketball team from a class of 30 students if,
a) Nobody can be on two teams.
b) Any number of students can be on both the teams.
c) At most one student can be on both the teams.

4. A student in physics needs to spend five days in a laboratory during her last semester of studies. After each day in the lab, she needs to spend at least six days in her office to analyse the data before she can return to the lab. After the last day in the lab, she needs ten days to complete her report that is due at the end of the last day of the semester. In how many ways can she do this if we assume that the semester is 105 days long?

5. In how many ways can the elements of $[n]$ be permuted so that the sum of every two consecutive elements in the permutation is odd?

6. Give a combinatorial proof of the following equations, i.e., prove that both sides are counting the same thing.

a) $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$

b) $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$

c) $2.\left(\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} + \binom{2n}{n}\right) = \binom{2n+2}{n+1}.$

d) $\sum_{j=0}^{k}\binom{n}{j} = \sum_{j=0}^{k}\binom{n-1-j}{k-j}2^j.$

7. Prove that $\frac{(2n)!}{n!^2}$ is even if n is a positive integer. Further prove that $\frac{(2n)!}{n!^2} < 5^n$.

8. Let $p = p_1 p_2 \ldots p_n$ be a permutation of $[n]$ and assume that $n \geq 3$. We say that i is an excedance of p if $p_i > i$. Compute the number of permutations of $[n]$ whose excedance set contains at least one of $n-2$ and $n-1$.

9. In how many ways can we list the digits $\{1,1,2,2,3,3,4,5\}$ so that two identical digits are not in consecutive positions?

10. Euler's totient function for a positive integer n, denoted by $\phi(n)$, is defined as the number of integers k in the range $1 \leq k \leq n$ for which the $gcd(n,k) = 1$. Prove that $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$.

Use principle of inclusion-exclusion. ($p \mid n$ denotes prime p divides n.)

Solutions

Solution 1

The total number of ways the agent can visit 4 cities 5 times without any restriction is the same as the different linear orderings of a multiset of 20 elements that contains 4 items with 5 copies of each, i.e., $\frac{20!}{5!5!5!5!}$. Now suppose the agent visits the cities in such a way that he starts and finishes with city 1. The number of different ways this can be done is $\frac{18!}{3!5!5!5!}$ as once he fixes city 1 in the first and the last position, he is free to visit the rest of cities in any order. Clearly, if he starts and finishes with any other city the number $\frac{18!}{3!5!5!5!}$ will remain the same. Now, to get our answer we can simply subtract the total number of “objects” with the number of “bad objects”. Therefore, the answer is $\frac{20!}{5!5!5!5!} - 4.\frac{18!}{3!5!5!5!}$.

Solution 2

For every couple there are 3 choices (i) no one gives a speech, (ii) 1st partner gives a speech, and (iii) 2nd partner gives a speech. Thus, total number of choices will be 3^n. This includes the case where no one makes a speech.

Solution 3

a) We can first pick soccer team in $\binom{30}{11}$ ways and then from the rest of the 19 players we can pick the basketball team in $\binom{19}{5}$ ways because no student from the soccer team must be there in the basketball team. So the total number of ways teams' selection can be done is $\binom{30}{11}\binom{19}{5}$.

b) Since there are no restrictions we can first choose the soccer team in $\binom{30}{11}$ ways and then the basketball team in $\binom{30}{5}$ ways. Therefore, the total number of ways will be $\binom{30}{11}\binom{30}{5}$.

c) First pick the student that is present in both the teams. This can be done in 30 ways. Then, from the rest of the 29 players we can pick the other 10 members of the soccer team in $\binom{29}{10}$ ways and then from the rest of the 19 players, we can pick other 4 members of basketball team in $\binom{19}{4}$ ways. So the total number of ways teams can be selected with exactly one common player is

$30\binom{29}{10}\binom{19}{4}$. Add $\binom{30}{11}\binom{30}{5}$ to this number as it is the number of ways teams can be selected with no common player.

Solution 4

Similar to a problem discussed in the class this problem again seems to be counting the number of subsets with some restriction. In particular, we need to find a subset $\{x_1, x_2, x_3, x_4, x_5\}$ of $[105]$ such that $x_5 \leq 95$ (because she needs 10 days to complete the report after the last lab) and $x_{i+1} > x_i + 6$, for $i \in [4]$ (because she needs at least 6 days in the office between two labs).

Based on these observations, we now give a bijection from set of 5 element subsets of $[71]$ to 5 element subsets of $[105]$ with the above mentioned restrictions. Following is the bijection.

$f(\{x_1, x_2, x_3, x_4, x_5\}) = \{x_1, x_2 + 6, x_3 + 12, x_4 + 18, x_5 + 24\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$

I leave the proof of why this is a bijection to you.

Solution 5

Observe that in order to make the sum of every two consecutive elements in the permutation odd, no to two consecutive elements can be of the same parities . The answer now depends on the parity of n.

When n is odd: The permutation cannot start from an even number because it will lead to two odd numbers in consecutive positions. The only possibility of arrangement in this case is (odd, even, odd, …, odd). The number of odd numbers in $[n]$ is $(n+1)/2$ and the number of even numbers in $[n]$ is $(n-1)/2$. It should be easy to see now that the total number of possible permutations will be $((n+1)/2)! \times ((n-1)/2)!$.

When n is even: The only possibility of arrangement in this case is (odd, even, odd, …, even) or (even, odd, even, …, odd). The number of odd numbers and even numbers in $[n]$ is $n/2$. Therefore, the number of arrangements that starts from an odd number is $(n/2)! \times (n/2)!$ and the the number of arrangements that starts from an even number is also $(n/2)! \times (n/2)!$. Hence, the total number of permutations are $2 \,.\, (n/2)! \times (n/2)!$.

Solution 6

a) Both sides are counting the number of n-size subsets of set $[2n]$. LHS is trivial so we will focus only on RHS. One way to select an n-size subset of $[2n]$ is to first divide the set $[2n]$ into two halves of size n. Then, we can pick i elements from first half and $n - i$ elements from the second half. Together this can be done in $\binom{n}{i} \cdot \binom{n}{n-i}$ ways. Now, possible values of i will range from 0 to n.

Therefore, in this manner, the total number of ways to pick n-size subsets from $[2n]$ will be
$\sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$.

b) Consider a set of n boys and n girls. Both sides are counting the number of ways we can select an n member team from set of these $2n$ people such that the captain of the team is a girl.

First modify LHS as $\sum_{k=1}^{n} k\binom{n}{k} \cdot \binom{n}{n-k}$. The number of ways to select a team with k girls and $n-k$ boys and a girl captain is $k \cdot \binom{n}{k} \cdot \binom{n}{n-k}$. The value of k ranges from 1 to n as we need to have a girl in the team since the captain is always a girl. Therefore, the sum will be
$\sum_{k=1}^{n} k\binom{n}{k} \cdot \binom{n}{n-k}$.

In RHS, we are first selecting a girl captain in n many ways and then the rest of $n-1$ players from $2n-1$ players in $\binom{2n-1}{n-1}$ ways. Hence, the answer is $n \cdot \binom{2n-1}{n-1}$.

c) Both sides are counting $n+1$ size subsets of $[2n+2]$. RHS is trivial. Let us focus on LHS.

Let us take two elements from $[2n+2]$ say x and y. Consider the following cases of selecting a subset of size $n+1$.

Case 1: x is present, but y is not.
In this case, we need to select other n many elements from the remaining $2n$ elements.

Case 2: y is present, but x is not.
Same as the previous case.

Case 3: Both x and y are not present.
In this case, we need to pick $n+1$ element from the remaining $2n$ elements. Let us divide the remaining $2n$ elements into two equal halves. Clearly, we will need to pick at least one element from both the halves as we have $n+1$ elements to pick. The number of ways to pick k elements from the first half and $n-k+1$ elements from the second half is $\binom{n}{k}\binom{n}{n+1-k}$, which is the same as $\binom{n}{k}\binom{n}{k-1}$. The value of k can range from 1 to n. Hence, the total number of ways to select a subset of size $n+1$ from $2n$ elements, in this manner, is $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}$.

Case 4: Both x and y are present.
In this case, we need to pick $n-1$ element from the remaining $2n$ elements. But this is the same as not-selecting $n+1$ elements from the remaining $2n$ elements. (Recall $\binom{n}{k} = \binom{n}{n-k}$). Hence the answer for this case is the same as the answer for previous case, which is $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}$.

Now if we sum the answers for the above 4 cases we will get LHS.

d) Both sides are counting the number of binary strings of length n with at most k many 1s. LHS is trivial, hence, we will focus only on RHS. Let $S =$ the set of binary strings of length n with at most k many 1s.

If there are at most k many 1s in a binary string, there will be at least $n-k$ many 0s in the binary string. The RHS counts the number of binary strings of length n with at most k many 1s with respect to the position of $(n-k)$th 0. Suppose the position of $(n-k)$th 0 is $(n-j)$. Then the number of binary strings in S where $(n-k)$th 0 is at $(n-j)$ position will be $\binom{n-j-1}{n-k-1}.2^j$ ($n-k-1$ many 0s to the left of $(n-k)$th 0 and any number of 0 to the right of $(n-k)$th 0). But $\binom{n-j-1}{n-k-1}.2^j = \binom{n-j-1}{k-j}.2^j$. Range of j will clearly be from 0 to k. Therefore, the sum will be $\sum_{j=0}^{k} \binom{n-j-1}{k-j}.2^j$.

Solution 7

$\frac{2n!}{n!n!} = \binom{2n}{n}$ is the number of subsets of size n from $[2n]$. Let S be set of subsets of size n from $[2n]$. Suppose every element of S is represented by a unique dot on the plane. Connect any two element of S, say x and y, with a line if x and y are complement of each other. Now, every element of S will be the endpoint of exactly one line as every element will have exactly one complement which is present in S. Since every line is connected to 2 elements, we can say that the total number of elements in S is $2k$, where k is the number of lines that we have drawn.

Another proof is $\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1} = 2.\binom{2n-1}{n-1}$.

Solution 9

First notice that, in a permutation p, $n-1$ can be the part of p's excedance set only if p has n at the $(n-1)$th place. Therefore, the number of permutations of $[n]$ with $n-1$ in their excedance set is $(n-1)!$ (put n at $n-1$ position and arrange other elements in $(n-1)!$ ways). Similarly, in a permutation p, $n-2$ can be the part of p's excedance set only if p has n or $n-1$ at the $(n-2)$th place. Hence, the number of the number of permutations of $[n]$ with $n-2$ in their excedance set will be $2.(n-1)!$. Now, in a permutation p, both $n-1$ and $n-2$ will be the part of p's excedance set only if p has n at the $(n-1)$th place and $n-1$ at the $(n-2)$th place. Therefore, the number of permutations of $[n]$ with $n-1$ and $n-2$ in their excedance set is $(n-2)!$.

Now we can apply PIE. Number of permutations with at least one of $n-1$ or $n-2$ in the excedance set = Number of permutations with $n-1$ in the excedance set + Number of permutations with $n-2$ in the excedance set − Number of permutations with $n-1$ and $n-2$ in the excedance set = $3.(n-1)! - (n-2)!$.

Solution 9

We will use PIE. The number of permutations of $\{1,1,2,2,3,3,4,5\}$ so that two identical digits are not in consecutive positions = Number of all permutations of $\{1,1,2,2,3,3,4,5\}$ − Number of permutations of $\{1,1,2,2,3,3,4,5\}$ where at least one pair of identical digits is in consecutive positions.

Now, number of all permutations of $\{1,1,2,2,3,3,4,5\} = \dfrac{8!}{2!2!2!1!1!}$. Let A_i be the set of permutations with both i in consecutive positions, for $i \in 1,2,3$. Then, $|A_i| = \dfrac{7!}{2!2!1!1!1!}$ (just tie is together and count all permutations). Also, $|A_{i_1} \cap A_{i_2}| = \dfrac{6!}{2!1!1!1!1!}$ and $|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 5!$.

Now, the number of permutations of $\{1,1,2,2,3,3,4,5\}$ so that two identical digits are not in consecutive positions $= \dfrac{8!}{2!2!2!1!1!} - (3.\dfrac{7!}{2!2!1!1!1!} - 3.\dfrac{6!}{2!1!1!1!1!} + 5!)$.

Solution 10

Let $\{p_1, p_2, \ldots, p_k\}$ be the set of prime divisors of n and $n = p_1^{a_1} . p_2^{a_2} . \ldots . p_k^{a_k}$.
Let $A_i = \{x \mid x \in [n], p_i \text{ divides } x\}$. Now, clearly $|A_i| = n/p_i$. In general, $|A_{i_1} \cap A_{i_2} \cap \ldots A_{i_j}| = \dfrac{n}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_j}}$.

Let us apply PIE now. Clearly, $\phi(n) = n - |A_1 \cup A_2 \cup \ldots \cup A_k| = n - \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le k}$

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j}| = n\left(1 - \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le k} \frac{1}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_j}}\right)$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_k}\right) = n\prod_{p|n}\left(1 - \frac{1}{p}\right).$$